Distributed active sonar detection in correlated $K$-distributed clutter

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Abstract—Distributed active sonar systems exploit the simultaneous detection of targets on multiple receiver platforms to improve system performance. False alarm performance modeling for such systems typically assumes independence from sensor to sensor; however, false alarms from clutter (e.g., shipwrecks, mud volcanoes, or rock outcappings) are expected to produce correlated data. In this paper, a clutter model exhibiting intersensor correlation and having $K$-distributed marginal probability density functions is proposed and analyzed. Under the constraint of equal sensor-level performance, the $m$-of-$n$ fusion processor was seen to be optimal for most cases of interest for all but the heaviest-tailed clutter. The system-level probability of false alarm is derived for the $m$-of-$n$ fusion processor and approximations are developed for the system-level probability of detection for common target models. In analyzing fusion performance, the AND processor ($m = n$) was seen to perform best in heavy clutter when the target model had some measure of inter-sensor consistency while $m/n \approx 0.25$ was seen to perform best for the highly-variable Gaussian target model. The importance of accounting for inter-sensor correlation in false alarm performance modeling was illustrated by an up to 10 dB over-estimation of performance when the clutter data are incorrectly assumed to be independent.

Index Terms—active sonar, distributed detection, clutter, $K$ distribution, data fusion

I. INTRODUCTION

Compared with a monostatic sonar system, the use of distributed sensing in active sonar can increase area coverage, improve detection performance, and refine localization by obtaining multiple, diverse observations of the target [1], [2], [3]. Distributed active sonar systems are typically designed so that a target is simultaneously detected on multiple sensors (despite variability in bistatic target strength [4] [5]) which can lead to improved localization and classification performance. However, false alarms arising from target-like clutter (e.g., shipwrecks, rock outcrops, or mud volcanoes [6], [7]) are also expected to be detected on multiple sensors. In this paper a statistical model is proposed to represent such spatially-correlated clutter and evaluated in the context of distributed detection where the data from individual sensors are fused to decide if a target is present or not.

Communications constraints and a desire for robustness to mismatch often dictate the use of a two-tiered detection network topology where only binary decisions from the individual sensors are used to form the system-level decision [8], [9]. The simplest analysis of this type of decentralized hypothesis testing involves modeling the data from multiple sensors as statistically independent which leads to the application of a likelihood-ratio detector at each individual sensor [8]. The sensor-level thresholds are chosen to optimize the system-level performance and are, in general, interdependent. However, under the assumption of equal performance at each sensor, separately for both the clutter-only and clutter-plus-target hypotheses (respectively, the null and alternative hypotheses), the optimal fusion rule is to decide a target is present when at least $m$ of the $n$ sensors individually detect the target [10]. The so-called $m$-of-$n$ processor also arises in the processing of multiple radar sub-pulses [11] where it is known as binary integration or double-threshold detection. Specific cases of the $m$-of-$n$ processor include the AND fusion rule ($m = n$) requiring all $n$ sensors to detect the target for a system-level detection to occur and the OR fusion rule ($m = 1$) where only one sensor must detect the target.

Analysis of the $m$-of-$n$ fusion rule has not been completely restricted to independent sensor data. For example, both correlated signals [12] and correlated noise [13], [14] have been considered, although there do not appear to be any general analytical results. An interesting finding from [13] for a constant target in additive, correlated Gaussian or Laplacian noise was a degradation in detection performance as the inter-sensor noise correlation increased. Similarly, the bulk of distributed-detection analysis has been performed for Gaussian data (or equivalently for Gaussian-derived processes such as Rayleigh-distributed envelope data). In addition to the examination of Laplacian noise in [13], other non-Gaussian-derived distributions considered include the Weibull and $K$ envelope distributions in [15] and a generalized Gaussian distribution in [10] and [16]. However, with the exception of [13], each assumed independence from sensor to sensor. Noting that the $m$-of-$n$ processor can be analyzed by considering the order statistics of the sensor-level detection statistics prior to
thresholding, the limited analysis for correlated data is most likely attributable to the general difficulty in analyzing the order statistics of correlated data [17].

The focus of this paper is on evaluating the performance of the \(m\)-of-\(n\) fusion processor in the presence of \(K\)-distributed clutter having inter-sensor dependence. In Section II, a model amenable to such analysis and having \(K\)-distributed marginal probability density functions (PDFs) is proposed and examined for representing correlated active sonar clutter echoes. The optimality of the \(m\)-of-\(n\) fusion processor is considered for this clutter model in Section III as well as deriving exact or approximate formulæ for the system-level false alarm and detection probabilities for non-fluctuating (herein referred to as Type 0) and Swerling Type I and II target models. Section IV examines the performance gain achieved by fusion in the presence of correlated \(K\)-distributed clutter, the effect of varying the clutter PDF tails, the optimal value of \(m\) in the \(m\)-of-\(n\) processor, and the error involved in incorrectly assuming inter-sensor clutter independence in sonar performance prediction.

II. MODELING OF CORRELATED ACTIVE SONAR CLUTTER

In a distributed active sonar system, a single-ping, system-level detection decision is typically made by combining data from range-bearing cells at each receiver pointing to the same geographic position (accounting for localization errors [19]). This is usually done in the context of centralized tracking [20], [21] with the sensor-level data as either the normalized matched filter intensity for the spatially overlapping range-bearing cells or, under severe communication constraints, binary sensor-level detection decisions. The majority of analysis of such systems assumes the sensor-level data to be statistically independent. Although this might be appropriate for false alarms from random threshold crossings arising from spatially homogenous processes such as diffuse reverberation or ambient noise, it is not expected to be accurate for false alarms arising from target-like echoes commonly called clutter in sonar (akin to clutter-discretes in radar), which can be a limiting factor in sonar system performance. Owing to the spatially compactness of many types of clutter sources (e.g., shipwrecks, mud volcanoes, rock outcroppings, etc.) the echoes measured at multiple sonar receivers are more likely to be correlated than statistically independent.

A similar situation occurs when fusing the data at a single sonar receiver from sub-pulses within a single-ping wavetrain [22] or from multiple consecutive pings (e.g., as in a track-before-detect scheme or tracklet formation in distributed tracking [23], [20]). The temporal stationarity of the ocean over short time periods leads to an expectation of correlated multi-ping or sub-pulse data. A similar assumption is made in the derivation of the Swerling Type I target model (see for example the discussion in [11, Sect. 9.2]) where the amplitude is taken as constant over several closely-spaced observations (e.g., waveform sub-pulses), but then changes randomly over longer times (from scan to scan). The viability of synthetic aperture sonar processing for low-frequency active sonar systems [24] and reverberation nulling using time-reversal [25] support this assumption for active sonar clutter as both require coherence or correlation between echoes on multiple, consecutive pings. The argument is weaker for distributed systems where sensors might receive echoes from different aspects. However, some correlation is still expected for spatially-compact clutter sources having azimuthally isotropic or broad scattering patterns.

Although the focus of this paper is on the detection performance of a distributed system when fusing data from a single ping, many of the concepts, results, and analysis are applicable to the alternative example of fusing active sonar data over multiple pings at a single receiver.

A. A real-data example

As an example of what correlation might be observed between clutter echoes, a preliminary analysis of data obtained from the NATO Undersea Research Centre’s Clutter 2007 experiment (please see the acknowledgments in Sect. VI) is presented in Fig. 1. The Clutter 2007 experiment took place in the Malta Plateau area of the Mediterranean Sea from early May to early June of 2007. The data presented in Fig. 1 were obtained from circular tracks 2 km distant from a shipwreck at 36°18.804′N by 14°44.136′E on May 12, 2007 and around a mud volcano field about the point 36°34.218′N by 14°25.878′E on May 21, 2007. A 0.5-s linear frequency modulated (LFM) waveform from 800–1800 Hz was transmitted every 12 s yielding a sampling of the clutter-source echoes at an aspect resolution of less than one degree. After matched filtering and beamforming, the clutter echo on each ping was isolated by manually choosing the beam with the highest signal-to-noise power ratio (SNR) and then through the use of Page’s test as described in [26] to find the echo start and stop times. The maximum correlation coefficient (see eq. (7) in Sect. II-E) between the intensity signals on each pair of echoes is computed. The average of all samples with aspect separation within 0.5 degrees of each unit degree from (−180, 179) is then plotted in Fig. 1. As seen in the figure, the shipwreck and mud volcano field show a correlation coefficient between 0.4 and 0.5 with larger values at the lower aspect separations and a slow fall off as the aspect separation grows. For comparison, a Swerling Type I target and independent Rayleigh and \(K\)-distributed (shape parameter \(\alpha_0 = 1\)) clutter were simulated with an echo duration of 200 independent samples (approximately the same as the real data) and subjected to the same processing. As derived in Sect. II-E, the Swerling Type I target with an SNR of 12.7 dB is expected to yield a correlation coefficient of 0.9. Although the independent clutter data should result in zero correlation, the process of choosing the maximum results in a biased estimate—it is therefore expected that the real data results are similarly biased high. However, clutter echoes from different aspects arising from the same source transmission might be expected to have a higher correlation than those arising from multiple monostatic transmission as is the case with these data. Note that at 0° aspect separation in the figure, the real data contain some values in (−0.5°, 0.5°) that were not equal to zero (the simulated data were all at 0° separation).
and therefore did not have a unit correlation coefficient, which causes the reduced average value seen in the figure. The data in Fig. 1 present some initial evidence of correlation for clutter echoes in a distributed active sonar system. A more detailed analysis of the measurements is currently underway.

**B. The K distribution**

Both empirical evidence [27], [28], [7] and theoretical models [18], [29], [27], [30] support the use of the K distribution [18] to model the statistical fluctuations of clutter echoes. The K-distribution PDF and cumulative distribution function (CDF) are [31]

\[
f_K(t; \alpha_0, \lambda_0) = \frac{2}{\lambda_0 \Gamma(\alpha_0)} \left( \frac{t}{\lambda_0} \right)^{\alpha_0 - 1} K_{\alpha_0 - 1} \left( 2 \sqrt{\frac{t}{\lambda_0}} \right) \tag{1}
\]

and

\[
F_K(t; \alpha_0, \lambda_0) = 1 - \frac{2}{\Gamma(\alpha_0)} \left( \frac{t}{\lambda_0} \right)^{\alpha_0} K_{\alpha_0} \left( 2 \sqrt{\frac{t}{\lambda_0}} \right) \tag{2}
\]

where \( t \geq 0 \) is the matched filter intensity, \( \alpha_0 \) is the shape parameter and \( \lambda_0 \) is a scale parameter. The shape parameter controls the tails of the PDF with large values producing Rayleigh-like envelope data and small values yielding heavy-tailed clutter.

Of the theoretical models resulting in the K distribution, the product form [30] is exploited in this paper to derive a model for correlated active sonar clutter echoes. The product form of the K distribution describes the matched filter intensity as the product

\[
T = UV \tag{3}
\]

where \( V \) is gamma distributed with shape parameter \( \alpha_0 \) and scale parameter \( 1/\alpha_0 \) and \( U \) is exponentially distributed with mean \( \sigma_U^2 = \alpha_0 \lambda_0 \) and independent of \( V \). In the radar sea-surface clutter community, \( U \) is the speckle component representative of Rayleigh backscattering (i.e., \( \sqrt{U} \) is Rayleigh distributed) and \( V \) is a modulation induced by wave motion with empirical evidence supporting it following the gamma distribution.

**C. A model for spatial dependence**

In general, correlated samples can be difficult to analyze owing to the need for a joint probability density function over all the variables. Typically, the multivariate Gaussian distribution is invoked in representing correlated data owing to the simplicity of decorrelating the data and the tractability of analysis after the data are whitened. In developing a model representing correlated active sonar clutter, the need for a tractable analysis must be balanced with the desire for an accurate representation of the scattering physics. As mentioned in the previous section, a model having the K distribution as the univariate marginal PDF is supported by both empirical analysis and theoretical models. The theoretical models with a physical basis [29], [27] do not readily lend themselves to a multiple-observation model with a tractable analysis. However, the product form of the K-distribution [30] does. Given \( n \) observations of a clutter source with each being marginally K-distributed, the matched filter intensity of the \( i \)th observation would have the form

\[
T_i = V_i U_i \tag{4}
\]

where \( V_i \) and \( U_i \) are independent and, respectively, gamma and exponentially distributed as described in Sect. II-B. This model has been used in the radar community [32], [33], [34] to represent spatially-correlated clutter; however, the correlation is in short time (contiguous delay cells on a given beam and ping) rather than across sensors or pings. In [32] and [33] correlation is induced by making each modulation pair \((V_i, V_j)\) dependent while in [34] each pair of speckle components \((U_i, U_j)\) is additionally made correlated.

A more appropriate model might be found from [35] where data from multi-look synthetic aperture radar (SAR) were modeled as K-distributed where \( V_i \) was held constant for all \( i \) and the speckle components \((U_i, U_j)\) were correlated. At the complex matched filter envelope stage, such a model is identical to the spherically invariant random vector (SIRV) models which have seen significant use in the radar community [36] and been exploited in the sonar community [37] to represent data in the neighborhood of a test cell across delay and beam. Unfortunately, when applied to fusing the data from multiple observations, these more general dependent clutter models are typically not easy or at times even possible to analyze.

A further simplification to the statistical model described by (4) is required to enable analysis of the m-of-n fusion processor. Following [35], let \( V_i \) be constant over all observations, but let the pair \((U_i, U_j)\) be independent for \( i \neq j \). The sensor-level detection statistic, including an additive target component, would have the form

\[
T_i = \left| \sqrt{V} U_i + A_i e^{2\pi \phi_i} \right|^2 \tag{5}
\]
for \( i = 1, \ldots, n \) sensors where the clutter component is represented by \( \sqrt{V} \tilde{U}_i \) and the target by the latter term which will be described in more detail below. When \( \tilde{U}_i \) is zero-mean, complex-Gaussian distributed with power \( \sigma_0^2 \) and \( V \) is gamma distributed with shape \( \alpha_0 \) and scale \( \alpha_0^{-1} \) (so \( E[V] = 1 \)), the clutter component is \( K \)-distributed. Assuming that the speckle components (\( \tilde{U}_i \)) are independent from observation to observation while holding \( V \) constant across the \( n \) observations but random from ping to ping (i.e., each new set of \( n \) observations) introduces dependence in the clutter while at the same time limiting the correlation. There is no strong physics-based rationale supporting this model; however, noting that the non-Rayleighness of the clutter in this model comes strictly from the randomness of \( V \) and working under the hypothesis that Rayleigh-distributed clutter should generally be independent across observations, it is reasonable for intersensor dependence to be modeled at this level. This model is not expected to represent all clutter with dependence across observations but to provide a means for analyzing distributed-system performance in the presence of spatially-dependent clutter.

Various target models (e.g., the Swerling models as described in \([11]\)) may be represented by the form \( A_i e^{j2\pi \phi_i} \) in (5). Typically, the phase is assumed to be random (i.e., \( \phi_i \) is uniformly distributed on \((0, 2\pi)\)) from observation to observation; however, this is not pertinent in the present analysis where data from each observation are combined incoherently. The non-fluctuating (herein referred to as a Type 0) target model, which provides the most consistency from observation to observation, simply requires a constant and non-random amplitude: \( A_i = A_0 \). The Swerling Type I model assumes the target has a constant amplitude over these \( n \) observations, but varies according to a Rayleigh distribution from ping to ping (i.e., \( A_i \) is Rayleigh distributed with power \( E[A_i^2] = \sigma_i^2 \)). The Gaussian fluctuating or Swerling Type II target model varies the most with a Rayleigh-distributed amplitude that is independent from observation to observation: \( A_i \sim \) independent and identically distributed (i.i.d) Rayleigh with power \( E[A_i^2] = \sigma_i^2 \).

\[
D. \text{ Independent clutter models}
\]

As a baseline for comparison with the dependent clutter model of (5), independent clutter models with data of the form

\[
T_i = \left| \sqrt{V_i} \tilde{U}_i + A_i e^{j2\pi \phi_i} \right|^2
\]

(6)

are utilized where \( V_i \) and \( \tilde{U}_i \) are independent within and across observations. Rayleigh-distributed clutter are easily obtained by setting \( V_i = 1 \). Independent \( K \)-distributed clutter is obtained when the \( V_i \) are i.i.d Gamma with shape \( \alpha_0 \) and scale \( \alpha_0^{-1} \). Note that as \( \alpha_0 \rightarrow \infty \) both the dependent and independent \( K \)-distributed clutter models converge to the independent Rayleigh clutter model.

\[
E. \text{ Observation-to-observation correlation}
\]

In a general sense, the clutter model described in (5) has more consistency between observations than the fluctuating target (Swerling Type II) and less than that of a non-fluctuating target (Type 0). To quantify this, consider the observation-to-observation correlation coefficient [38],

\[
\rho = \frac{E[T_i T_j] - \mu_i \mu_j}{\sigma_i \sigma_j}
\]

(7)

where \( i \neq j, \mu_i = E[T_i], \) and \( \sigma_i^2 = \text{Var}[T_i] \). When no target is present, it can be shown that \( \rho \) is

\[
\rho = \frac{\sigma_V^2}{1 + 2\sigma_V^2}
\]

(8)

where \( \sigma_V^2 = \text{Var}[V] \).

From this it can be seen that when \( \sigma_V^2 \rightarrow 0 \), the inter-observation correlation goes to zero while increasing \( \sigma_V^2 \) results in a correlation no higher than \( 1/2 \). When clutter is \( K \)-distributed, the inter-observation correlation is

\[
\rho_K = \frac{1}{2 + \alpha_0}
\]

(9)

illustrating an inverse relationship between the correlation and shape parameter which indicates that heavy-tailed clutter will have a high correlation that decreases as the clutter tends to the Rayleigh distribution.

The Swerling Type I model in Rayleigh clutter results in an inter-observation correlation of

\[
\rho_1 = \frac{1}{\left(1 + \frac{\sigma_V^2}{\sigma_i^2}\right)^2}
\]

(10)

which tends to one as the signal-to-clutter power ratio (SCR = \( \sigma_i^2/\sigma_V^2 \)) increases. Note that an SCR of only 3.8 dB is required to provide more correlation than any clutter model of the form found in (5) while an SCR of 12.7 dB yields an inter-observation correlation of 0.9.

III. DISTRIBUTED-SYSTEM \( P_{fa} \) AND \( P_d \)

particularly in distributed systems, deciding whether a target is present or not from multiple observations is often done only knowing if a target was detected by an individual sensor or not but without knowing the precise value of the sensor-level detection statistic (\( T_i \)). Define the sensor-level decisions as

\[
Q_i = U(T_i - h)
\]

(11)

for \( i = 1, \ldots, n \) where \( h \) is the sensor-level detector threshold and \( U(x) \) is the unit step function so that \( Q_i = 1 \) when a detection occurs on the \( i \)th sensor (i.e., \( T_i \geq h \)) and is zero otherwise. The \( m \)-of-\( n \) fusion processor declares a system-level detection when the sum of the binary data is greater than or equal to \( m \),

\[
\hat{Q} = \sum_{i=1}^{n} Q_i \geq m.
\]

(12)

The sensor-level data may be modeled individually as Bernoulli\((p)\) random variables where \( p \) is one minus the CDF of the sensor-level detection statistic evaluated at the sensor-level detector threshold. When both the clutter and target statistics are independent and identically distributed over the \( n \) observations, the system-level detection statistic \( \hat{Q} \) is a
binomial\((n, p)\) random variable. Therefore, from [11, Sect. 9.8], the system-level exceedance distribution function (EDF) is
\[
P_e(H_j) = \Pr \{ \bar{Q} \geq m | H_j \} = 1 - \sum_{i=0}^{m-1} \binom{n}{i} F_T^{n-i}(h; H_j) [1 - F_T(h; H_j)]^i \tag{13}\]
from which \(P_{fa}\) is obtained under the clutter-only hypothesis \((H_0)\) and \(P_d\) under the clutter-plus-target hypothesis \((H_1)\). In (13) \(F_T(h; H_j)\) is the CDF of \(T_i\) under hypothesis \(H_j\).

Dependent clutter and the Swerling Type I target model clearly violate the above assumptions. However, owing to the commonality of \(V\) in the clutter component in (5) across all sensors, \(P_{fa}\) and some \(P_d\) analysis are feasible for the \(m\)-of-\(n\) fusion processor. In the following sections, conditions for the optimality of the \(m\)-of-\(n\) processor in dependent \(K\)-distributed clutter are described, an analytical result for the system-level \(P_{fa}\) is derived, and then approximations and numerical evaluations of \(P_d\) for the various target models are presented.

\section{Optimality of \(m\)-of-\(n\) fusion in spatially-correlated clutter}

In [10], Reibman and Nolte showed that under a restricted scenario where the sensor-level data are statistically independent with constant \(P_d\) and \(P_{fa}\) across all sensors, the \(m\)-of-\(n\) fusion processor implements the likelihood ratio of binary data describing sensor-level alarms and is therefore optimal. Unfortunately, the result does not universally extend to the dependent clutter model described in (5). However, it will be shown that the \(m\)-of-\(n\) processor is optimal for many cases of interest.

Conditioning on \(V\) in (5) results in independent sensor data under \(H_0\) and under \(H_1\) for the Type 0 and Swerling Type II targets. Extension to the Swerling Type I target will be described at the end of the section. Define the sensor-level \(P_d\) and \(P_{fa}\) conditioned on \(V\) as \(p_d(V)\) and \(p_{fa}(V)\) where the lower-case \(p\) denotes sensor-level probabilities rather than generic or system-level. These sensor-level conditional detection measures are simply derived from the standard sonar/radar data describing sensor-level alarms and is therefore optimal. Under the clutter-plus-target hypothesis \((H_1)\) and \(\bar{q}\) is a sufficient statistic, the likelihood ratio, when described as a function of \(\bar{q}\), is
\[
l(\bar{q}) = \frac{f_1(\bar{q})}{f_0(\bar{q})} = \int_0^\infty \frac{p_d(v)^\bar{q} [1 - p_d(v)]^{n-\bar{q}} f_V(v) dv}{\int_0^\infty p_{fa}(v)^\bar{q} [1 - p_{fa}(v)]^{n-\bar{q}} f_V(v) dv}.	ag{19}\]
If \(l(\bar{q})\) is monotonically increasing with \(\bar{q}\), then comparing \(Q\) to a threshold (as is done in the \(m\)-of-\(n\) fusion processor) implements the likelihood ratio test on the binary sensor-level detection data and provides the highest system-level \(P_d\) for a fixed \(P_{fa}\).

Unfortunately, \(l(\bar{q})\) is not necessarily monotonic for all clutter and target distributions resulting in data of the form found in (5) and therefore the likelihood ratio of (19). Some indication of when it is comes from the following analysis. First (19) is described as a ratio of expectations
\[
E[\frac{g_d(V, \bar{q})}{g_{fa}(U, \bar{q})}] < E[\frac{g_d(V, \bar{q} + 1)}{g_{fa}(U, \bar{q} + 1)}], \tag{23}\]
for \(\bar{q} = 0, \ldots, n - 1\). Cross-multiplying in (23), subtracting by the right-hand side, and collating the arguments of the expectations under one expectation over both \(V\) and \(U\) results in the need to show that \(I < 0\) where
\[
I = E[\frac{g_d(V, \bar{q})g_{fa}(U, \bar{q} + 1) - g_d(V, \bar{q} + 1)g_{fa}(U, \bar{q})}{g_{fa}(U, \bar{q})}]. \tag{24}\]
By exploiting the mean-value theorem for double integrals [40] extended to apply to expectations (e.g., see [41, Sect. 12.111] for the univariate case), a point \((u_0, v_0) \in \mathbb{R}^2\), where \(\mathbb{R}^+ = [0, \infty)\), can be found such that
\[
I = g_d(v_0, \bar{q}) g_{fa}(u_0, \bar{q} + 1) - g_d(v_0, \bar{q} + 1) g_{fa}(u_0, \bar{q}). \tag{25}\]
Owing to the discrete nature of \(Q\), a randomized test [39] might be required to achieve certain values of \(P_{fa}\).
It is then straightforward to show that $I < 0$ when

$$p_{fa}(u_0) < p_d(v_0).$$

(26)

Note that the pair $(u_0, v_0)$ will depend on $\bar{q}$, so (26) must hold for all $\bar{q} = 0, \ldots, n - 1$.

Recalling that $E[V] = E[U] = 1$, it is reasonable to expect that $u_0$ and $v_0$ are near one, with less deviation when $\sigma_v^2$ is small, which represents the case of near-Rayleigh clutter and weak inter-sensor correlation. Noting that $p_d(v) > p_f(v)$ for all $v$ for any non-zero additive signal, and that $p_d(v)$ and $p_f(v)$ are increasing functions with $v$ implies that (26) will be satisfied whenever $u_0 < v_0$. Additionally, when the SCR is high enough to result in $p_d(1) \gg p_f(1)$ (most cases of interest), the monotonicity will likely persist for many values of $u_0 > v_0$. However, when the PDF of $V$ has significant weight at high values, $u_0$ might be too large resulting in a non-monotonic $l(\bar{q})$.

A numerical analysis of the monotonicity of the likelihood ratio with $\bar{q}$ for correlated, $K$-distributed clutter and the Type 0 and Swerling Type II targets confirmed the expectation that increasing SCR and decreasing clutter tail heaviness encourages monotonicity and also identified a dependence on $n$ where if $l(\bar{q})$ is monotonic for a given $n$ it is also monotonic for smaller values of $n$—perhaps arising from a greater reduction in $v_0$ than $u_0$ as $n$ increases. The results of the numerical analysis are therefore shown in Fig. 2 as the maximum value of $n$ for which $l(\bar{q})$ is monotonic as a function of $\alpha_0$ for various values of SCR, the Type 0 and Swerling Type II targets, and a sensor-level $P_{fa} = 10^{-4}$. The likelihood ratio for the Type 0 target is seen to exhibit monotonicity at very low values of SCR ($\leq 5$ dB) in the presence of very heavy-tailed clutter ($\alpha_0 \leq 1$) for moderate values of $n$. The sensor-level $P_d$ for the Type 0 target was less than $10^{-3}$, even for the largest SCR (5 dB) and lightest clutter tails ($\alpha_0 = 1$) shown in the figure. The Swerling Type II target likelihood ratio required a higher level of SCR for monotonicity in the heavy-tailed clutter; for the 15 dB SCR case, the sensor-level $P_{fa}$ was 0.27 at $\alpha_0 = 0.5$ and 0.42 at $\alpha_0 = 1$. However, differing from the Type 0 target, the likelihood ratio was always found to be monotonic when $\alpha_0 > 1$. Lowering the sensor-level $P_{fa}$ shifted the curves for the Type 0 target to the left and the Swerling Type II target to the lower right (respecting the vertical asymptote at $\alpha_0 = 1$), while increasing the sensor-level $P_{fa}$ had the opposite effect. Thus, the $m$-of-$n$ processor is foreseeably the optimal processor for correlated, $K$-distributed clutter in most situations of interest, with some counter examples in the combination of very heavy-tailed clutter, large $n$, and SCR not high enough for moderate sensor-level $P_d$.

The above analysis may be extended to account for a Swerling Type I target or, more generally, any target with a random amplitude that is constant over all sensors. This is accomplished by including the random target amplitude ($A$) in the expectation in the numerator of (20). The inequality providing monotonicity then becomes

$$p_{fa}(u_0) < p_d(v_0, \alpha_0)$$

(27)

where $\alpha_0$ is obtained from the mean-value theorem. Similar results to those reported for the Type 0 and Swerling Type II targets are expected for the Swerling Type I target.

B. $P_{fa}$ analysis

As previously mentioned, the most common analysis of the $m$-of-$n$ fusion rule follows from characterizing the sensor-level data as being iid Bernoulli random variables. Alternatively, the detection statistic can be taken as the $(n-m+1)$st order statistic of the sensor data $\{T_1, \ldots, T_n\}$. Let the ordered detection statistics be

$$T_{(1)} \leq T_{(2)} \leq \cdots \leq T_{(n)}. \quad (28)$$

Thus if $T_{(n-m+1)}$ exceeds the sensor-level detection threshold, then at least $m$ of the $n$ samples exceed that threshold. Though the two implementations of the $m$-of-$n$ fusion processor are identical in terms of their statistical performance, the order-statistic-based implementation is less preferable in practice as it requires access to the detection statistic from each sensor rather than the binary, single-sensor decision.

In general, it is difficult to analyze the ordered statistics of correlated data [17]. However, by conditioning on $V$, the data are independent under $H_0$ and follow an exponential distribution for which previous results may be exploited to determine the system-level $P_{fa}$ of the dependent clutter model in (5). Conditioned on $V$ under $H_0$, the sensor-level detection statistics $\{T_1, \ldots, T_n\}$ are iid exponential random variables with mean $\lambda = V\sigma_v^2$. The $(n-m+1)$st order statistic may be described as

$$T_{(n-m+1)} = \sum_{i=1}^{n-m+1} \Delta_i$$

(29)

where $\Delta_i$ is the difference between the $i$th and $(i-1)$st order statistics (i.e., $\Delta_i = T_{(i)} - T_{(i-1)}$) with $\Delta_1 = T_{(1)}$. From [42, Chapt. 1.6], when the data are iid exponential random variables with mean $\lambda$, $\Delta_1, \ldots, \Delta_n$ are independent and
exponentially distributed with mean $E[\Delta_i] = \lambda/(n - i + 1)$. The characteristic function of $T_{(n-m+1)}$ may then be formed as the product of the characteristic functions of the $\Delta_i$ and simplified to

$$\Phi(\omega) = \prod_{i=1}^{n-m+1} \left( 1 - \frac{j\omega \lambda}{n - i + 1} \right)^{-1} \hspace{1cm} (30)$$

$$= \sum_{i=1}^{n-m+1} b_{n-i+1} \left( 1 - \frac{j\omega \lambda}{n - i + 1} \right)^{-1} \hspace{1cm} (31)$$

where the summation in (31) is obtained through partial fraction expansion which yields the coefficients

$$b_i = \prod_{l=m, l \neq i}^{n} \left( 1 - \frac{i}{1} \right)^{-1} \hspace{1cm} (32)$$

The PDF of $T_{(n-m+1)}$ conditioned on $V$ is then easily seen to be

$$f_0(t|V) = \sum_{i=m}^{n} b_i \frac{e^{-t/(\lambda/i)}}{\lambda/i} \hspace{1cm} (33)$$

which is a mixture of exponential PDFs, each with mean $\lambda/i = V\sigma_0^2/i$. Recalling that for $K$-distributed clutter data conditioned on $V$, $T_i$ is exponentially distributed with mean $V\sigma_0^2$, it is seen that each term in the sum of (33) will result in a $K$-distributed intensity PDF $(f_K(t; \alpha, \lambda)$ from (1)) when the conditioning on $V$ is removed with the scale parameter scaled by $i^{-1}$,

$$f_0(t) = E_V[f_0(t|V)] = \sum_{i=m}^{n} b_i f_K \left(t; \alpha_0, \frac{\sigma_0^2}{i\alpha_0} \right); \hspace{1cm} (34)$$

that is, the PDF of the $(n-m+1)$st order statistic for the dependent clutter model is a mixture of $K$-distribution PDFs. Integrating the PDF of the $(n-m+1)$st order statistic results in the CDF which is a mixture of $K$-distribution CDFs ($F_K(t; \alpha, \lambda)$ from (2)),

$$F_0(t) = \sum_{i=m}^{n} b_i F_K \left(t; \alpha_0, \frac{\sigma_0^2}{i\alpha_0} \right). \hspace{1cm} (35)$$

The system-level $P_{fa}$ for the dependent clutter model of (5) is then easily obtained from (35) as

$$P_{fa} = 1 - F_0(h)$$

$$= 1 - \sum_{i=m}^{n} b_i F_K \left(h; \alpha_0, \frac{\sigma_0^2}{i\alpha_0} \right)$$

$$= \frac{2}{\Gamma(\alpha_0)} \left( \frac{h\alpha_0}{\sigma_0^2} \right)^{\alpha_0} \sum_{i=m}^{n} b_i \left( i \right)^{\alpha_0} K_{\alpha_0} \left( \frac{2}{\sigma_0^2} \sqrt{hi\alpha_0} \right) \hspace{1cm} (36)$$

where $h$ is the sensor-level detection threshold and the fact that

$$\sum_{i=m}^{n} b_i = 1 \hspace{1cm} (37)$$

has been exploited.

C. Extension of $P_{fa}$ analysis to other clutter models

Owing to the form of (33), these results are not restricted to clutter following the $K$-distribution, but apply to any model that may be described as a common amplitude modulating an iid Gaussian speckle component. For example, if the clutter envelope followed a Rayleigh-mixture distribution with probability $p_i$ of observing power $\pi_i$ for $l = 1, \ldots, L$ (which yields average clutter intensity $\sigma_0^2 = \sum_{i=1}^{L} \pi_i \gamma_i$), then the $P_{fa}$ is

$$P_{fa} = \sum_{i=m}^{n} \sum_{l=1}^{L} b_i \pi_i e^{-hi/\gamma_i}. \hspace{1cm} (38)$$

Similarly, if the clutter intensity data are Weibull distributed with shape parameter $\beta$ and average intensity $\sigma_0^2 = \gamma \Gamma(1 + \beta^{-1})$, then

$$P_{fa} = \sum_{i=m}^{n} b_i \gamma e^{-(hi/\gamma)^{\beta}} \hspace{1cm} (39)$$

where $\beta = 1$ results in independent Rayleigh clutter.

D. $P_{d}$ evaluation

In this section methods are provided for evaluating the system-level $P_{d}$ for each of the target models being considered. Approximations are developed that enable $P_{d}$ computation using at most a one-dimensional numerical integral over standard functions. The approximations are compared with simulation results for a variety of scenarios to determine their region of validity. The values considered for each parameter in the comparison are shown in Table I. For all cases considered, $\sigma_0^2 = 1$ and the SCR is varied. For the Type 0 target model, $\text{SCR} = A_0^2/\sigma_0^2$ while for the Swerling Type I and II models it is $\sigma_0^2/\sigma_0^2$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Values evaluated</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>3, 7, and 11</td>
</tr>
<tr>
<td>$m$</td>
<td>1, $(n+1)/2$, and $n$</td>
</tr>
<tr>
<td>$\alpha_0$</td>
<td>1, 2, 10, and 100</td>
</tr>
<tr>
<td>$P_{fa}$</td>
<td>$10^{-2}, 10^{-4}$, and $10^{-6}$</td>
</tr>
<tr>
<td>$\sigma_0^2$</td>
<td>1</td>
</tr>
<tr>
<td>SCR</td>
<td>various</td>
</tr>
</tbody>
</table>

### 1) Type 0 target:

The Type 0 or non-fluctuating target results in sensor data (from (5) with $A_1 = A_0$)

$$T_i = \left| \sqrt{V} \hat{U}_i + A_0 e^{j2\pi\phi_i} \right|^2 \hspace{1cm} (40)$$

where the target amplitude $A_0$ is deterministic (i.e., constant and known). As previously noted, $T_i$ given $V$ is the scale of a non-central chi-squared random variable with two degrees of freedom. Thus, the system-level $P_{d}$ can be described by inserting the non-central chi-squared CDF ($F_{\chi^2_a}(x)$) into (13) to obtain $P_{d}$ for the $m$-of-$n$ processor given $V$ followed by
an expectation over $V$ which removes the aforementioned conditioning, resulting in

$$P_d(h) = 1 - \int_0^\infty f_V(v) \times \sum_{i=0}^{m-1} \binom{n}{i} \frac{\alpha^{i-1} e^{-\alpha v}}{(\Gamma(\alpha))} \bigg[ 1 - F_{\chi^2_s} \left( \frac{2h}{v \sigma_0^2} \right) \bigg]^i \, dv$$

where the non-centrality parameter $\delta = 2A_0^2 / (v \sigma_0^2)$ is a function of $v$. As $V$ is gamma distributed with unit mean and variance $1/\alpha_0$, it has PDF

$$f_V(v) = \frac{\alpha_0^\alpha v^{\alpha-1}}{\Gamma(\alpha_0)} e^{-\alpha_0 v}$$

and the integral can be limited to a region around one that collapses as $\alpha_0$ increases. The non-central chi-squared CDF is related to Marcum’s Q-function [43] and is itself described by a one-dimensional integral or infinite summation. If a subroutine for evaluating Marcum’s Q function or the non-central chi-squared CDF is not available or if computational capability is limited, many approximations may be found in [44] including Pearson’s shifted-gamma and Sankaran’s power-law-normal approximations. These approximations are fairly accurate except for low values of the CDF which essentially equates to low threshold values or high $P_{fa}$. Sankaran’s most flexible approximation is detailed in the Appendix and used in (41) in a comparison with simulation to determine its accuracy. An example of the results may be found in Figure 3 for $P_{fa} = 10^{-4}$, $n = 7$ and $\alpha_0 = 1$. A total of 5,000 trials were used in the simulation. The average and maximum absolute errors in SCR over all of the scenarios described in Table I for $P_d \in [0.25, 0.95]$ are shown in Table II where it can be seen that both are fractions of a decibel for the Type 0 target. As might be expected from using Sankaran’s approximation to the non-central chi-squared CDF, accuracy was worst for the highest $P_{fa}$ and $n$ and lowest $m$ and SCR.

### Table II

<table>
<thead>
<tr>
<th>$P_{fa}$</th>
<th>Type 0</th>
<th>Swerling Type I</th>
<th>Swerling Type II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-4}$</td>
<td>0.037 (0.18)</td>
<td>0.19 (0.95)</td>
<td>0.075 (0.45)</td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td>0.027 (0.16)</td>
<td>0.14 (0.55)</td>
<td>0.055 (0.43)</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>0.018 (0.099)</td>
<td>0.12 (0.57)</td>
<td>0.06 (0.48)</td>
</tr>
</tbody>
</table>

2) Swerling Type I target: The Swerling Type I target, as utilized here to represent target echoes received on multiple sensors in a distributed active sonar system, fluctuates randomly from ping to ping, but is constant within a ping resulting in sensor data (from (5) with $A_i = A$)

$$T_i = \left| \sqrt{V} \widetilde{U}_i + A e^{j2\pi \phi} \right|^2$$

where the target amplitude $A$ is a Rayleigh-distributed random variable with power $E[A^2] = \sigma_t^2$. The system-level $P_d$ can be determined exactly by conditioning on both $A$ and $V$, inserting the non-central chi-squared CDF in (13), and then removing the conditioning by a two-dimensional integral implementing the expectation over $A$ and $V$. However, an accurate and less computationally expensive approximation is obtained by assuming that the clutter envelope is Rayleigh distributed and independent from sensor to sensor (i.e., let $\alpha_0 \rightarrow \infty$ or $V = 1$) resulting in

$$P_d(h) \approx 1 - \int_0^\infty f_A(a) \sum_{i=0}^{m-1} \binom{n}{i} \frac{\alpha^{i-1} e^{-\alpha a}}{(\Gamma(\alpha))} \bigg[ 1 - F_{\chi^2_s} \left( \frac{2h}{\sigma_0^2} \right) \bigg]^i \, da$$

where $\delta = 2a^2 / \sigma_0^2$ and

$$f_A(a) = \frac{2a}{\sigma_t^2} e^{-a^2 / \sigma_t^2}.$$

Additional computational savings are obtained by using Sankaran’s approximation to the non-central chi-squared CDF as described in the previous section and the Appendix. At the expense of more complicated coding, the computational burden could be reduced further through a technique described by Walker [45] (which can be found in [46, Sect. 3.4]) where the region of integration over the target amplitude is limited by exploiting the fact that the sum in (44) tends to one for large enough values of $a$.

The iid-Rayleigh-distributed clutter approximation subverts one of the integrals and leads to significant computational savings, yet surprisingly results in an accurate approximation, as can be seen in Fig. 3 and Table II. Of the three target types, this approximation results in the highest average errors, although they are still less than one fifth of a decibel. The worst case maximum SCR error was just under one decibel and occurred for the highest $P_{fa}$ (owing to Sankaran’s approximation), the lowest $\alpha_0$ (owing to the Rayleigh clutter approximation), and the highest $n$ and $m$. As illustrated in Fig. 3, the approximation slightly underestimates performance for $m = n$ but is quite accurate for lower values of $m$. 

Fig. 3. Modeled and simulated $P_d$ for the three target models for $n = 7$, $m = 1, 4$, and 7, $P_{fa} = 10^{-4}$, and $\alpha_0 = 1$. The model results are solid lines while the simulated results are dashed.
3) Swerling Type II target: The Swerling Type II target provides the most fluctuation of the three targets considered with independent random fluctuation from sensor to sensor. The sensor data are as in (5) with the $A_1$, $A_2$, $A_3$, iid Rayleigh distributed with power $\sigma_a^2$. The system-level $P_d$ can be determined exactly by conditioning on $V$, evaluating (13) with an exponential CDF with mean $E[T_i|V] = V\sigma_0^2 + \sigma_a^2$ followed by an integral removing the conditioning on $V$. However, as with the Swerling Type I target of the previous section, an approximation assuming iid Rayleigh-distributed clutter provides an accurate and easily evaluated alternative,

$$P_d(h) \approx 1 - \sum_{i=0}^{m-1} \frac{n^i}{i!} \left[ 1 - e^{-h/(\sigma_a^2+\sigma_0^2)} \right]^{n-i} e^{-h/(\sigma_a^2+\sigma_0^2)}. \quad (46)$$

As seen in Fig. 3 and Table II, the average absolute error in SCR is less than one tenth of a decibel for each $P_{fa}$ and the maximum errors all less than half of a decibel.

It is surprising that assuming iid Rayleigh-distributed clutter leads to an accurate approximation for the Swerling Type I and II target models while it fails to provide an accurate approximation for a non-fluctuating target. Heuristically, it might be assumed that at high SCR values the target component dominates the clutter and therefore drives performance; however, the approximation appears accurate at lower values of SCR and yet does not work well for the non-fluctuating target even at high SCR. An alternative explanation lies in the randomness of the target amplitude dwarfing the additional randomness induced by the gamma-distributed $\sqrt{V}$ modulating iid Rayleigh clutter. The non-fluctuating target model has no randomness in amplitude and is therefore more sensitive to the randomness induced by the $\sqrt{V}$ factor on the clutter. This explanation also supports the better fit for the Swerling Type II target observed in the model-simulation evaluation compared with that for the Type I target.

The significance of this result is that it enables the use of standard techniques for evaluating the system-level $P_d$ for the two Swerling target models for the correlated clutter model described in (5) and is probably applicable to more clutter PDFs than the $K$ distribution.

IV. DISTRIBUTED-SYSTEM DETECTION PERFORMANCE ANALYSIS

The $P_{fa}$ formula and $P_d$ approximations developed in the previous section are utilized in this section to evaluate the system-level performance of the $m$-of-$n$ fusion processor in the dependent clutter model.

A. Fusion gain

The use of multiple sensors in distributed detection is intended to increase system-level performance by increasing coverage and/or system-level $P_d$ for a fixed $P_{fa}$. This improvement in the $P_d$-$P_{fa}$ fusion receiver operating characteristic (ROC) curves is illustrated in Figs. 4–6 for the three target models. In each figure, the OR processor ($m = 1$), AND processor ($m = n$), and median processor ($m = (n + 1)/2$) are evaluated for increasing numbers of sensors ($n = 1, 3, 7, 11, \text{ and } 21$), heavy clutter ($a_0 = 1$), and moderate levels of SCR. In Fig. 4 it is seen that the AND processor provides the best performance for the Type 0 target at low values of $P_{fa}$ while the OR processor provides little or no performance gain ($P_d$ is only increasing with $n$ when the sensor-level $P_d$ is very high). The median processor provides the best performance at moderate values of $P_{fa}$ (e.g., above about $5 \times 10^{-5}$). As illustrated here and in the following section, the best value of $m$ depends on many factors.

The fusion ROC curves shown in Fig. 5 for the Swerling Type I target are similar to those for the Type 0 target although the median processor is only better than the AND processor for $P_{fa}$ above about $10^{-2}$. The improvement in system-level $P_d$ for a fixed $P_{fa}$ for the AND and median processors is also not as significant for the Swerling Type I target compared with that observed for the Type 0 target in Fig. 4.

In Fig. 6 the fusion ROC curves for the Swerling Type II target show that the median processor provides the best performance for larger values of $n$ (11 and 21) while the OR processor is best for smaller values of $n$. The AND processor provided no fusion gain regardless of the number of sensors—note that all the AND fusion ROC curves overlay each other—yielding an identical result to the case of a fluctuating target in independent Rayleigh-distributed clutter. This may be explained by relating the sensor-level threshold for fusing $n$ sensors ($h_n$) to that for fusing one ($h_1$) and then showing that, for a fixed the system-level $P_{fa}$, the system-level $P_d$ does not change with $n$. Conditioning on $V$, the sensor data are iid exponentially distributed with mean $\lambda = V\sigma_0^2 + \sigma_a^2$ and the system-level detection statistic for the AND processor is $T_{(1)}$, the minimum order statistic. The minimum of $n$ iid exponential random variables with mean $\lambda$ is also exponentially distributed but with mean $\lambda/n$. Thus, the system level threshold may be obtained by inverting

$$P_{fa} = E_V \left[ \text{Pr} \left\{ \frac{U V \sigma_0^2}{n} > h_n | V \right\} \right] \quad (47)$$

for $h_n$, where $U$ is exponentially distributed with unit mean. From (47) it is clear that the threshold for combining $n$ sensors is related to that for just one sensor according to $h_n = h_1/n$. Substituting this into the system-level $P_d$,

$$P_d(n) = E_V \left[ \text{Pr} \left\{ \frac{U (V\sigma_0^2 + \sigma_a^2)}{n} > h_n | V \right\} \right] = E_V \left[ \text{Pr} \left\{ \frac{U V \sigma_0^2 + \sigma_a^2}{n} > h_1 | V \right\} \right] = P_d(1), \quad (48)$$

illustrates the invariance of detection performance with $n$. The distribution of $V$ in (47) and (48) will change the specific values of $h_n$ and $P_d(n)$, but does not alter the relationships $h_n = h_1/n$ and $P_d(n) = P_d(1)$ indicating that no fusion gain may be obtained for a Swerling Type II target using the AND processor irrespective of whether the clutter data are Rayleigh, $K$, or some other distribution as long as the data follow the correlated clutter model described by (5).
Fig. 4. Fusion curves for Type 0 target in heavy-tailed clutter for the AND, OR, and median processors for \( n = 1, 3, 7, 11, \) and 21. The \( n = 1 \) curves are the same for each processor. The AND processor performs best at the lowest \( P_{fa} \) values, the median processor is best above approximately \( P_{fa} = 5 \times 10^{-5} \), and the OR processor provides little or no performance improvement.

Fig. 5. Fusion curves for Swerling Type I target in heavy-tailed clutter for the AND, OR, and median processors for \( n = 1, 3, 7, 11, \) and 21. The \( n = 1 \) curves are the same for each processor. The AND processor provides no fusion gain, the median processor provides better performance than the OR processor for the larger values of \( n \), but not the smaller values where the OR processor is best.

Fig. 6. Fusion curves for Swerling Type II target in heavy-tailed clutter for the AND, OR, and median processors for \( n = 1, 3, 7, 11, \) and 21. The \( n = 1 \) curves are the same for each processor. The AND processor provides best performance at the lowest \( P_{fa} \) values, the median processor is best above \( P_{fa} = 10^{-2} \), and the OR processor provides little or no performance improvement.

**B. Optimal \( m \) of \( n \) and varying clutter tails**

As seen in the previous section, the value of \( m \) providing the best performance varies with target type, \( n \), and \( P_{fa} \). To examine this variation, the SCR required to achieve a specific system-level performance measure (\( P_{fa} = 10^{-6} \) and \( P_{d} = 0.5 \)) is evaluated for all \( m \) for various values of \( n \) (ranging from \( n = 2 \) to 20), heavy-tailed clutter (\( \alpha_0 = 1 \)), and for each target type. In Fig. 7 it is seen for the Type 0 target that for all values of \( n \) considered the AND processor (\( m = n \)) results in the least amount of SCR required to achieve the performance specification. The disparity in performance from \( m = 1 \) (the worst case) to \( m = n \) varied from about 3 dB at \( n = 2 \) to 10 dB at \( n = 20 \). Similar results are seen for the Swerling Type I target in Fig. 8 excepting the \( n = 20 \) case where the optimal value of \( m \) is 19. The Swerling Type II target, however, had varying values of \( m \) yielding the best performance as seen in Fig. 9. When \( n \leq 4 \), the OR processor (\( m = 1 \)) provided the best performance while the ratio \( m/n \) tended toward 0.25 as \( n \) increased.

Unfortunately, these results do not appear to generalize too much further. The fusion ROC curves seen in Fig. 10 illustrate how performance changes when the clutter varies from heavy-
targets while finding the optimal value of \( m/n \) tended to remain relatively constant with minor variation from small to large values of \( n \). Under the assumption that the errors will be expected, the performance degrades as the clutter becomes more tailed (and correlated) to Rayleigh (and independent) with \( \alpha_0 = 1, 2, 10, \text{ and } \infty \). The \( n = 20 \) case is considered where \( m = 12 \) for both the Type 0 and Swerling Type I targets while \( m = 6 \) for the Swerling Type II model. As expected, the performance degrades as the clutter becomes heavier tailed with the Swerling Type II target most affected and the Swerling Type I target least affected. The values of \( m \) in this example were chosen because they optimize performance (in terms of minimizing the SCR required to achieve a performance specification) for independent Rayleigh clutter (\( \alpha_0 = \infty \)) as shown on the right side of Fig. 11.

The optimal value of \( m \) for \( n = 20 \) is shown as a function of \( \alpha_0 \) for the three target types in Fig. 11 when \( P_{fa} = 10^{-6} \) and \( P_d = 0.5 \). For the Type 0 and Swerling Type I targets, the optimal value of \( m \) decreased from \( n \) to approximately \( n/2 \) as \( \alpha_0 \) increased. Thus, for targets with some measure of consistency, the AND processor is best in correlated, heavy-tailed clutter shifting down to a median processor in independent Rayleigh-distributed clutter. For the highly fluctuating Swerling Type II target, the optimal value of \( m \) stayed relatively constant with minor variation from \( m/n \) equal 0.25 to 0.3 as the clutter ranged from heavy-tailed and correlated to Rayleigh and independent.

Examining Figs. 7–9, it can be seen that choosing

\[
m = \text{round} \left( n \psi(\alpha_0) \right)
\]

(49)

where \( \psi(\alpha_0) \in (0, 1) \) depends only on the target type and \( \alpha_0 \) can provide nearly optimal performance. For example, for \( \alpha_0 = 1 \), choosing \( \psi(1) = 1 \) for the Type 0 target or \( \psi(1) = 0.97 \) for the Swerling Type I result in the optimal value of \( m \) for all the values of \( n \) evaluated. For the Swerling Type II target, choosing \( \psi(1) = 0.25 \) results in \( m \) being off by one for two values of \( n \). Under the assumption that the errors will be small if \( \psi(\alpha_0) \) is chosen from a large enough value of \( n \) and understanding that one will generally not know \( \alpha_0 \) precisely, using the values found in Fig. 11 will result in nearly optimal performance for a wide range of scenarios. As seen in the figure, the curves for the Type 0 and Swerling Type I targets may be reasonably approximated by

\[
\psi(\alpha_0) \approx \max \{ 0.5, \min \{ 1, 0.98 - 0.21 \log_{10} \alpha_0 \} \}.
\]

(50)

These results change slightly with the \( P_dP_{fa} \) specification. For example, increasing \( P_{fa} \) to \( 10^{-4} \) resulted in a slightly more rapid descent to the median for the consistent targets and a slower transition from \( m/n = 0.25 \) to 0.3 for the Swerling Type II target as \( \alpha_0 \) increased. Increasing \( P_d \) to 0.9 had a...
similar effect, though it was more pronounced for the Type 0 target and $m/n = 0.25$ was optimal for the Swerling Type II target for all $\alpha_0$.

![Image](image_url)

Fig. 11. Optimal value of $m/n$ for the performance specification of $P_{fa} = 10^{-6}$ and $P_d = 0.5$ for $n = 20$ as a function of $\alpha_0$. Targets with some measure of consistency prefer the AND processor in heavy clutter and transition to the median processor in Rayleigh clutter while the Swerling Type II target had minimal variation over $\alpha_0$ from $m/n = 0.25$ to 0.3.

C. Performance prediction

In the absence of knowledge about the dependence of clutter from sensor to sensor, it is natural to assume independence as previously described in (6) and, for example, as in [10], [16], [15]. However, as will be shown in this section, such an assumption can lead to significant overestimation of distributed sonar system performance when the clutter is in fact dependent across sensors.

First note that for the Swerling Type I and II targets, the $P_d$ approximations developed in Sect. III-D assumed the clutter was iid Rayleigh. As such, it is not surprising when these approximations work equally well for the independent $K$-distributed clutter model. When compared with $P_d$ obtained through simulation (the same as that described in Sect. III-D except using independent $K$-distributed clutter for each sensor), the models resulted in similar average absolute SCR error values to those shown in Table II. The Type 0 model, however, does rely significantly on the characterization of the clutter. Interestingly, the system-level $P_d$ for this target model in dependent $K$-distributed clutter, as a function of threshold, was found to be very similar to that for the independent $K$-distributed clutter model. The average absolute SCR error against simulation was less than one tenth of a decibel for $P_{fa} = 10^{-4}$ and $10^{-6}$ and about one seventh of a decibel for $P_{fa} = 10^{-2}$. The greatest disparity occurred when $P_{fa}$ and $n$ were high and $P_d$, $\alpha_0$, and $m$ were low. As such, assuming $P_d(h)$ is the same for the two clutter models is reasonable for all three target models, particularly for low $P_{fa}$ and low to moderate values of $n$.

Owing to the monotonicity and negative slope of $P_d$ in threshold $h$, it is therefore only necessary to show that the threshold for a given $P_{fa}$ for dependent $K$-distributed clutter ($h_d$) is greater than the threshold for independent $K$-distributed clutter ($h_i$) to illustrate that assuming independence overestimates performance (i.e., if $h_d \geq h_i$, then the predicted $P_d(h_i) \geq P_d(h_d)$). The threshold ratio when $P_{fa} = 10^{-6}$ is shown in Fig. 12 as a function of $m$ for even values of $n$ ranging from 2 to 20 and various values of $\alpha_0$. When $\alpha_0$ is large, the thresholds are very similar as expected from the low correlation in the dependent $K$ clutter model (recall $\rho_K = 1/(2 + \alpha_0)$). However, for small values of $\alpha_0$ the thresholds can differ significantly—even more than an order of magnitude for very heavy-tailed clutter ($\alpha_0 = 0.5$). Unfortunately, one of the limitations of the correlated clutter model of (5) is that the inter-sensor correlation is coupled with the clutter tail heaviness. As such, it is possible that near-Rayleigh clutter (i.e., large $\alpha_0$) with a higher inter-sensor correlation than that afforded by the model of (5) may result in a more significant prediction disparity than that shown in Fig. 12.

To relate the change in threshold to a change in performance, consider the formula for $P_d$ for the Swerling Type II target from (46) which is a function of $h/(\alpha_0^2 + \gamma^2)$. At high SCR, it is approximately a function of $h/\text{SCR}$ implying that a change in threshold will result in a commensurate change in the SCR required to maintain the same performance. The $P_d$ formulae for the Type 0 and Swerling Type I targets ((41) and (44)) do not immediately lead to any intuition on their functionality with respect to $h$ and SCR. However, numerical evaluation of the change in SCR required to account for a change in threshold, where the initial threshold and SCR are chosen to meet $P_{fa} = 10^{-6}$ and $P_d = 0.5$, showed a strong linearity at low $\alpha_0$ in units of decibels (i.e., a change in threshold of $\Delta h$ decibels required an increase in SCR of $\gamma \Delta h$ decibels to maintain $P_d$). The values of $\gamma$ ranged from around 0.8 when $m = n$ to 1.1 when $m = 1$ for $\alpha_0 = 1$ and $n = 11$. The range of $\gamma$ increased with both $\alpha_0$ and $n$. Thus, an increase in threshold equates to an approximately proportionate decrease in performance in terms of SCR required to meet a $P_d$ specification irrespective of the target model.

It is interesting to note that the thresholds in Fig. 12 appear identical when $m = 1$ for all values of $\alpha_0$ and $n$. The $P_{fa}$ for the dependent $K$-distributed clutter model shown in (36) is clearly different from that for the independent $K$-distributed clutter model as would be obtained by inserting the $K$-distribution CDF into (13). The phenomenon arises from a convergence of the $P_{fa}$ for the two models at high threshold values (they diverge at lower threshold values) and may be explained intuitively by arguing that because only one of the $n$ sensors must have a detection statistic exceeding a threshold, any correlation with the others should have minimal impact on the system-level $P_{fa}$. Alternatively, it is possible that conditioning on $V$ in the dependent clutter model followed by applying extreme-value theory [17] and an expectation over $V$ results in the same asymptotic distribution of the maximum order statistic as for independent $K$-distributed clutter.
Although the dependent clutter model proposed in this paper has no clear physical rational for its derivation and a limited capacity for correlation (inter-sensor intensity correlation for false alarms is always less than 0.5), the importance of accounting for inter-sensor correlation in false alarm performance modeling was emphasized by an up to 10-dB difference in system performance when data are incorrectly assumed to be independent. The disparity lessened as the clutter became more Rayleigh-like, although the effect in this model is confounded by a coupled lessening of the correlation.

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As described in [44, eq. 29.61d], Sankaran approximated a non-central chi-squared random variable with a normal variate raised to a power. The resulting CDF approximation is

\[ F_{\chi^2_{\nu,\delta}}(x) \approx \Phi \left( \frac{1}{\sqrt{2}} \left[ \frac{x - h_z}{\nu + \delta} \right] \right) \]

where \( \nu \) is the degrees-of-freedom parameter, \( \delta \) the non-centrality parameter, \( \Phi(z) \) is the standard-normal CDF,

\[ h_z = 1 - \frac{2}{3}(\nu + \delta)(\nu + 3\delta)(\nu + 2\delta)^{-2}, \]

\[ \mu_z = 1 + h_z(h_z - 1)(\nu + 2\delta)(\nu + 3\delta)(\nu + 2\delta)^{-2}, \]

\[ -h_z(h_z - 1)(h_z - 2)(3h_z - 1)(\nu + 2\delta)(\nu + 3\delta)(\nu + 2\delta)^{-2} \]

and

\[ \sigma_z^2 = 2h_z^2(\nu + 2\delta)(\nu + 3\delta)(\nu + 2\delta)^{-2} \]

\[ \left[ 1 - (h_z - 1)(3h_z - 1)(\nu + 2\delta)(\nu + 3\delta)(\nu + 2\delta)^{-2} \right]. \]